

Ratchet Effect Driven by Coulomb Friction: the Asymmetric Rayleigh Piston

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The effect of Coulomb friction is studied in the framework of collisional ratchets. It turns out that the average drift of these devices can be expressed as the combination of a term related to the lack of equipartition between the probe and the surrounding bath, and a term featuring the average frictional force. We illustrate this general result in the asymmetric Rayleigh piston, showing how Coulomb friction can induce a ratchet effect in a Brownian particle in contact with an equilibrium bath. An explicit analytical expression for the average velocity of the piston is obtained in the rare collision limit. Numerical simulations support the analytical findings.

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Introduction.— The problem of extracting work from unbiased noise is a central issue in the context of energy harvesting at small scales [1–3]. The phenomenon of the rectification of non-equilibrium fluctuations is called “ratchet effect”, from the seminal works of Smoluchowski [4] and Feynman [5], and is studied in the theory of Brownian motors [6–8]. This phenomenon can be achieved in the presence of dissipation, i.e. under statistical non-equilibrium conditions. It requires that both temporal and spatial symmetries are broken. Whereas these two fundamental constraints have been pointed out [5], the specific mechanisms ruling the action of ratchet devices in their different realizations still deserve a thorough study within general theories [9, 10].

A class of ratchet models is represented by the “collisional ratchets”, where fluctuations are induced by the interaction of an asymmetric probe with one (or more) gas(es) of particles. In these models, realized in several variants [11–17], dissipation is introduced in two ways: i) the probe is in contact with several baths at different temperatures; ii) interactions between the bath particles and the probe are dissipative (granular ratchets). The rectification of thermal fluctuations has also been observed in numerical simulations of an asymmetric particle diffusing in a glass forming liquid [18]. In this case, the energy flux sustaining the motor effect is induced by the thermal unbalance between fast and slow degrees of freedom.

Here we focus our attention on a different source of dissipation, introduced by the presence of Coulomb friction affecting the dynamics of the probe (or tracer) between successive collisions. This kind of dissipation modifies the dynamics of the tracer [19–22], and also plays an important role in the context of ratchet models [23–28]. Here we bring to the fore the structural elements common to general collisional ratchets when several forms of dissipation are present. We consider a general stochastic model ruling the dynamics of the probe’s velocity, where the interactions with the gas particles are described by a Master Equation and the presence of Coulomb friction is introduced through a deterministic force. For this model we derive an expression for the average drift where two terms appear: the first one takes into account the lack of equipartition due to non-equilibrium conditions, while

the second one is directly related to the frictional force. The first term represents the heat flux exchanged by the probe with the particles bath: it can be induced by a non-equilibrium coupling with the bath (e.g. if dissipative interactions or *reservoirs* at different temperatures are present), and/or by frictional dissipation.

To illustrate this result, we study the asymmetric Rayleigh piston [29] in the presence of Coulomb friction. In this model a Brownian tracer interacts with two gases of particles with different masses but at the same temperature. We show that the dissipation through friction, coupled with the spatial asymmetry introduced by different masses of gas molecules, is sufficient to induce a net drift on the piston, even if the gases are in equilibrium at the same temperature. We stress that the motor effect observed in our model originates from a mechanism different from those acting in systems where the piston is in contact with gases at different temperatures [12, 30], or with gas particles with different restitution coefficients [16, 26]: in all these systems, indeed, non-equilibrium currents are already present even in the absence of friction.

General model.— We consider a tracer of mass M and velocity V , the motion of which is constrained in one dimension, in contact with small particles of one or more gases characterized by different parameters: mass m_i , temperature T_i , density ρ_i , restitution coefficient r_i for the collisions between tracer and gas molecules ($r_i \in [0, 1]$, with $r_i = 1$ for elastic collisions), where the index i denotes different gases. Between successive collisions with the gas particles the dynamics of the tracer is affected by Coulomb friction, slowing its motion. Furthermore, we assume that the whole system, tracer plus gas(es), presents a spatial asymmetry.

The dynamical evolution of the system is represented by a piecewise deterministic process, described by stochastic jumps, modeling the interactions of the tracer with the particles, and by a deterministic term, taking into account the friction. The differential equation describing the probability density function of the tracer ve-

locity $P(V, t)$ is

$$\frac{\partial P(V, t)}{\partial t} = \int dV' [W(V|V')P(V', t) - W(V'|V)P(V, t)] + \Delta \frac{\partial}{\partial V} \sigma(V)P(V, t), \quad (1)$$

where Δ is the frictional coefficient, $\sigma(x)$ is the sign function and $W(V'|V)$ are the transition rates for the jump from V to V' due to collisions, which depend on the mass M and the gas parameters (ρ_i , T_i , m_i and r_i). The spatial asymmetry may appear in the model through the following structure of the transition rates

$$W(V'|V) = \begin{cases} W^+(V'|V) & \text{if } V' > V \\ W^-(V'|V) & \text{if } V' < V, \end{cases} \quad (2)$$

with $W^+(V'|V) \neq W^-(V'|V)$. This structure is realized for instance by the probe with triangular shape considered in [14, 15], where collisions on one side always accelerate the object whereas collisions on the other sides slow its motion. The condition (2) is not sufficient for a ratchet effect to be observed. In particular, if $\Delta = 0$ and the transition rates are assumed to satisfy the detailed balance (DB) relation with respect to an equilibrium distribution $P_0(V)$, then, denoting by $\langle \dots \rangle_0$ the average over $P_0(V)$, one has $\langle V \rangle_0 = 0$ and $\langle V^2 \rangle_0 = T/M$, where T is the temperature of the thermal bath of particles (if several gases are present, their temperatures have to be the same, i.e. $T_i = T$ for every i).

We stress that model (1) is more general than models described by Langevin equations (i.e. with white noise), widely studied in the literature of Brownian motors [7, 8]. Indeed, in a Langevin equation non-equilibrium conditions can only appear through time-varying parameters (potentials or temperatures), or through external forces. Here, the “kinetic” nature of noise is more physical and, due to the different time scales involved, gives the possibility to introduce non-equilibrium conditions such as constant different temperatures or other dissipation channels.

In this model it is important to put in evidence the competition between different timescales: the mean collision times which, in the limit of large mass M take the form $\tau_i \simeq \sqrt{m_i/T_i}/(\rho_i S_i)$, with S_i the scattering cross section of the tracer with the particles of gas i , and the stopping time $\tau_\Delta = V^*/\Delta$ due to friction, where V^* is the average velocity after a collision. These characteristic times introduce two regimes in the dynamics. For $\min\{\tau_i\} \gg \tau_\Delta$, the system is in the rare collision limit, where collisions always occur when the tracer is at rest because of friction. In this case, the dynamics evolves via slip-stick motions and the stationary distribution develops a singular contribution in $V = 0$, as explained in [26]. In the opposite limit, when $\max\{\tau_i\} \ll \tau_\Delta$, the system is in the frequent collision limit, and the tracer is never at rest.

Ratchet effect and lack of equipartition– To obtain a general expression for the average velocity of the tracer,

we multiply by V both members of Eq. (1) and integrate over V . In the stationary state, we get for the momentum flow

$$0 = -\Delta \langle \sigma(V) \rangle + \langle \alpha(V) \rangle, \quad (3)$$

where $\alpha(V) = \int (V' - V)W(V'|V)dV'$ is the jump moment, which depends on M through the rates W , and the symbol $\langle \dots \rangle$ denotes an average over the stationary distribution $P(V)$.

For mass of the tracer large enough with respect to the largest mass among those of the gas particles, denoted hereafter by m , we can do an expansion around $V = 0$ [31]. Keeping terms up to the second order, we obtain

$$\alpha + \alpha' \langle V \rangle + \frac{1}{2} \alpha'' \langle V^2 \rangle - \Delta \langle \sigma(V) \rangle \simeq 0, \quad (4)$$

where α' and α'' denote the first and second derivatives of α with respect to V , respectively, and all coefficients are computed in $V = 0$. In particular, $|\alpha'|^{-1}$ represents the characteristic thermalization time τ_{th} of the tracer with the gas, in the absence of friction. This time scale is related to the collision time: $\tau_{th} \sim \tau_i M/m_i$ [31]. The coefficients α , α' and α'' are functions of M through the transition rates, and have to be expanded in powers of M^{-1} consistently, taking into account that $\langle V^2 \rangle \sim \mathcal{O}(M^{-1})$. Eq. (4) yields

$$\begin{aligned} \langle V \rangle &= -\frac{1}{\alpha'} \left[\alpha + \frac{1}{2} \alpha'' \langle V^2 \rangle \right] + \frac{\Delta}{\alpha'} \langle \sigma(V) \rangle \\ &= -\frac{A}{\alpha'} [T_k - T] + \frac{\Delta}{\alpha'} \langle \sigma(V) \rangle, \end{aligned} \quad (5)$$

where in the second line we have assumed that thermal gradients (if present) are small so that one can define a base temperature T , and we have introduced the kinetic temperature $T_k \equiv M \langle V^2 \rangle$ (assuming $\langle V \rangle^2 \ll \langle V^2 \rangle$) and a general asymmetry A through the expressions

$$\alpha \simeq -TA, \quad \alpha'' \langle V^2 \rangle \simeq 2AT_k. \quad (6)$$

The above structure for the coefficients α and α'' is verified in many examples [11, 13, 14, 16, 17] including the one discussed below, and follows from Eq. (2) [32] (the explicit form of A depends on the specific model).

The interest of Eq. (5) is in making clear that there are two contributions to the ratchet’s drift, corresponding to the two channels for heat exchanges of the probe: the first one is $D_{hf} \equiv -\frac{A}{\alpha'} [T_k - T]$, which is proportional to the temperature difference $T_k - T$, and therefore to the heat flux exchanged between the ratchet and the thermal bath, induced by collisions; the second one is directly related to the presence of friction and is proportional to the average of the frictional force: $D_\Delta \equiv \Delta \frac{M}{\alpha'} \langle \sigma(V) \rangle$. Notice that the first channel can be sustained by the presence of *reservoirs* at different temperatures or dissipative collisions, but it is also affected by the presence of friction. Indeed, if elastic interactions are considered and all the baths are at the same temperature, a net flow can still be generated by frictional dissipation.

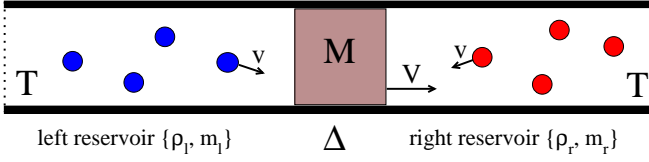


FIG. 1. The asymmetric Rayleigh piston with Coulomb friction.

For $\Delta = 0$, or when the thermalization time $\tau_{th} \sim 1/|\alpha'|$ is small with respect to the stopping time τ_Δ and friction can be neglected, only the term D_{hf} remains. Then the ratchet effect can be present if and only if the transition rates are asymmetric (i.e. $A \neq 0$) and do not satisfy DB, so that the kinetic temperature T_k is different from that of the external bath T . This is the case for many collisional ratchets studied in the literature [11, 13, 14, 16, 17], where the explicit expressions obtained for the drift in the different cases can be put in a form analogous to the first term of Eq. (5). In the opposite regime, namely when friction dominates, both channels are active and the two contributions may produce interesting interplays, with non-monotonic behaviors in the drift, as shown below. An expression reproducing the non-monotonic drift for collisional ratchets (which is a feature already observed, e.g. in [28]) represents a relevant result of the present study.

The asymmetric Rayleigh piston—To make more explicit our discussion, we now consider the generalized Rayleigh piston [29, 33], in the presence of Coulomb friction. It consists of a piston of mass M , the two faces of which are connected with two different gases of elastic particles of mass m_r (at right) and m_l (at left), see Fig. 1. The two gases are at equilibrium at the same temperature T and have densities $\rho_i = \rho$, with $i = r, l$. In such a way the pressures on both sides of the piston are equal. The piston velocity is changed by the elastic collisions with the (right and left) gas particles according to the rule $V' = V + \frac{2}{1+M/m_i}(v - V)$, where V and V' are the piston velocities before and after the collision, respectively. The particles velocities are distributed according to the Maxwell-Boltzmann distribution $p_i(v) = \rho_i \sqrt{\frac{m_i}{2\pi T}} \exp\left(-\frac{m_i v^2}{2T}\right)$, where the Boltzmann's constant $k_B = 1$.

In this model the asymmetric transition rates are [29]

$$\begin{aligned} W^+(V'|V) &= \left(\frac{M+m_l}{2m_l}\right)^2 (V' - V) \\ &\quad \times p_l\left(\frac{M+m_l}{2m_l}V' - \frac{M-m_l}{2m_l}V\right), \\ W^-(V'|V) &= \left(\frac{M+m_r}{2m_r}\right)^2 (V - V') \\ &\quad \times p_r\left(\frac{M+m_r}{2m_r}V' - \frac{M-m_r}{2m_r}V\right). \end{aligned} \quad (7)$$

These transition rates satisfy DB with re-

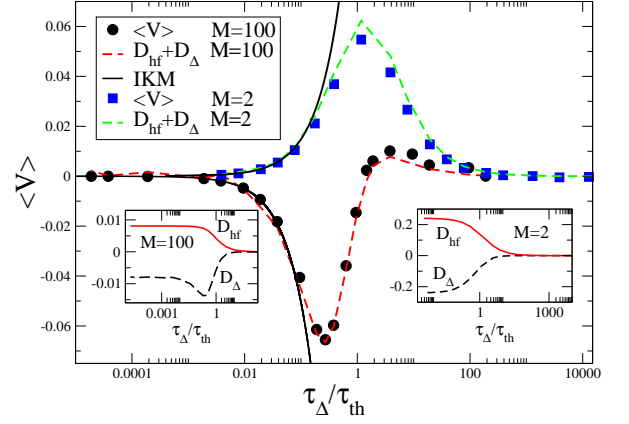


FIG. 2. (color online) DSMC results for $\langle V \rangle$, with $T = 10$, $\rho = 0.5$, $m_r = 2$ and $m_l = 1$, as a function of τ_Δ/τ_{th} , for $M = 100$ (black dots) and $M = 2$ (blue squares). The black curves show the analytical predictions of the IKM, Eq. (12). The dashed lines (red for $M = 100$ and green for $M = 2$) represent the rhs of Eq. (5) with $\langle V^2 \rangle$ and $\langle \sigma(V) \rangle$ computed in DSMC. The amplitude of $\langle V \rangle$ for $M = 100$ has been magnified by a factor 10, for scale reason. Insets: D_{hf} (red curves) and D_Δ (black dashed curves) vs τ_Δ/τ_{th} .

spect to the Gaussian distribution $P_0(V) = (2\pi T/M)^{-1/2} \exp(-MV^2/2T)$. The explicit expressions for the coefficients appearing in Eq. (5) are [29]:

$$\begin{aligned} \alpha &= \rho T[(M+m_l)^{-1} - (M+m_r)^{-1}] \\ &= -\rho T(m_l - m_r)/M^2 + \mathcal{O}(M^{-3}), \end{aligned} \quad (8)$$

$$\alpha' = -2\rho \sqrt{\frac{2T}{\pi}} \left[\frac{\sqrt{m_l}}{M+m_l} + \frac{\sqrt{m_r}}{M+m_r} \right], \quad (9)$$

$$\begin{aligned} \alpha'' &= 2\rho [m_l/(1+m_l/M) - m_r/(1+m_r/M)]/M \\ &= 2\rho(m_l - m_r)/M + \mathcal{O}(M^{-2}). \end{aligned} \quad (10)$$

From Eqs. (8) and (10) follows that the explicit formula for the asymmetry is $A \simeq \rho(m_l - m_r)/M^2$, which justifies the relations (6). A similar structure for the coefficients α and α'' can be traced back in many collisional ratchets [11, 13, 14, 16, 17]. In this model the time scales are $\tau_\Delta = V^*/\Delta = \sqrt{T/M}/\Delta$, because collisions are elastic, and $\tau_{th} = 1/|\alpha'| \simeq \sqrt{\pi/(2T)}M/[2\rho(\sqrt{m_l} + \sqrt{m_r})]$.

To study the behavior of the model and to verify the relation (5) in all regimes, we perform numerical simulations of the process (1) with transition rates (7), using a Direct Simulation Monte Carlo (DSMC) algorithm [34]. We extract the velocity v of a gas particle from $p_i(v)$, $i = r, l$ with probability 1/2, and then we allow the collision with the piston with velocity V to occur with probability $\propto |v - V|$. In Fig. 2 $\langle V \rangle$ is shown (black dots for $M = 100$ and blue squares for $M = 2$) as a function of the ratio τ_Δ/τ_{th} , which is varied by changing Δ , with the other parameters fixed (see caption). A net drift is found in a wide range of Δ values: we stress that, at variance with kinetic models studied previously, in this system the ratchet effect is entirely driven by the Coulomb friction,

because the two gases are in equilibrium at the same temperature and collisions are elastic.

The complex non-monotonic behavior of the drift is very well described in all the regimes by the r.h.s. of Eq. (5), represented in Fig. 2 by red (for $M = 100$) and green (for $M = 2$) dashed lines. The parameters α, α' and α'' are given by Eqs. (8), (9) and (10) and the averages $\langle V^2 \rangle$ and $\langle \sigma(V) \rangle$ are computed in DSMC. The behavior of the two terms D_{hf} and D_Δ is reported in the insets of Fig. 2. Both terms display plateaux in both the opposite limits $\tau_\Delta \ll \tau_{th}$ and $\tau_\Delta \gg \tau_{th}$. The plateau in the latter limit is zero for both terms, as equilibration with the thermal bath is quickly attained, inducing a zero drift. Since also in the opposite limit of rare collisions the drift is expected to vanish, as shown below, this produces the peaks observed in Fig. 2. Notice that in this model Eq. (5) also holds for values of M comparable to those of the gas particles (see Fig. 2), if all orders in M are retained in expressions (8), (9) and (10). This is due to the specific forms of the coefficients: in particular, all even derivatives of $\alpha(V)$ greater than the second one vanish for this model [29].

Independent kick model— An analytical explicit formula for the average drift can be obtained in the physical situation of rare collisions, namely when $\min\{\tau_i\} \gg \tau_\Delta$. In this case, assuming that every collision occurs when the piston is at rest, the average velocity can be computed in the so-called Independent Kick Model (IKM) [25, 26]. For our model this yields

$$\langle V \rangle = \left(\int dv |v| p_r(v) + \int dv |v| p_l(v) \right) \int_0^\tau V(t) dt, \quad (11)$$

where $V(t) = V_0 - \Delta \sigma(V_0)t$, $\tau = |V_0|/\Delta$ and V_0 is the velocity after a collision: $V_0 = V^+$ if $v > 0$, and $V_0 = V^-$ if $v < 0$, where $V^+ = \frac{2v}{1+M/m_r}$ and $V^- = \frac{2v}{1+M/m_l}$. Using these expressions one obtains

$$\langle V \rangle = \frac{2\rho}{\Delta} \sqrt{\frac{2T^3}{\pi}} \left[\frac{\sqrt{m_l}}{(m_l + M)^2} - \frac{\sqrt{m_r}}{(m_r + M)^2} \right]. \quad (12)$$

In this formula the net drift explicitly appears when the asymmetry in the system is present (i.e. $m_r \neq m_l$). For small Δ the formula is not expected to hold because the approximation of rare collisions is not valid. Notice also that in the limit $M \rightarrow \infty$ the drift vanishes. In Fig. 2, the analytical prediction (12) of the IKM (black lines) is shown to be in perfect agreement with the numerical results in the rare collision regime. Fig. 2 also shows that formula (5) is in agreement with the IKM prediction.

Conclusions— We have presented two interesting results: i) formula (5) for the average drift of a general collision ratchet in the presence of friction can describe the ratchet behavior in all regimes, and explicitly shows the two channels of dissipation contributing to the drift; this relation has been also tested in a rotor ratchet with dry friction recently studied in [28]; ii) Coulomb friction can be a source of dissipation sufficient to generate a ratchet effect in thermal baths. Our study can be extended to other forms of non-linear friction [35, 36].

Our results on the ratchet effect driven by Coulomb friction in a thermal bath pave the way to applications in the field of nanophysics. At these scales, thermal fluctuations can be induced by a gas of molecules or a liquid environment, and the Coulomb friction is still present, as well known from atomic friction experiments [37]. Moreover, the developments of new techniques for the design and fabrication of nano-devices can provide probes with desired shapes and asymmetries. Therefore, all the ingredients are available to realize ratchet devices at small scales entirely based on the action of Coulomb friction as source of dissipation.

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